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Nonequilibrium dynamics of a stochastic model of anomalous heat transport

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Abstract

We study the dynamics of covariances in a chain of harmonic oscillators with conservative noise in contact with two stochastic Langevin heat baths. The noise amounts to random collisions between nearest-neighbour oscillators that exchange their momenta. In a recent paper (Lepri *et al* 2009 *J. Phys. A: Math. Theor.* **42** 025001), we have studied the stationary state of this system with fixed boundary conditions, finding analytical exact expressions for the temperature profile and the heat current in the thermodynamic (continuum) limit. In this paper, we extend the analysis to the evolution of the covariance matrix and to generic boundary conditions. Our main purpose is to construct a hydrodynamic description of the relaxation to the stationary state, starting from the exact equations governing the evolution of the correlation matrix. We identify and adiabatically eliminate the fast variables, arriving at a continuity equation for the temperature profile $T(y, t)$, complemented by an ordinary equation that accounts for the evolution in the bulk. Altogether, we find that the evolution of $T(y, t)$ is the result of fractional diffusion.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Heat transport in lattices of nonlinear oscillators is a relevant test bed for understanding the behaviour of systems steadily kept out of equilibrium. The importance of this physical setup is further strengthened by the possibility of comparing theoretical predictions with experimental

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results, as indicated by the recent success in measuring heat conduction properties of individual nanotubes [1].

On the theoretical side, the chain of harmonic oscillators maintained out of equilibrium by means of stochastic heat reservoirs, is one of the few systems for which the nonequilibrium invariant state has been obtained rigorously [2]. However, due to the lack of any phonon scattering mechanism, its heat conductivity κ diverges linearly with the size of the chain. Consequently, Fourier's law of heat conduction $J_Q = -\kappa \nabla T$, relating the heat flux J_Q with the imposed temperature gradient ∇T does not hold. As a matter of fact, the integrability of the harmonic chain makes it impossible for the system to support a temperature gradient. It was already recognized by Debye that the presence of nonlinearities in the dynamics is a necessary condition for the occurrence of a *normal transport*, i.e. a finite heat conductivity in the thermodynamic limit. However, years later Fermi, Pasta and Ulam found that nonlinear dynamics does not necessarily induce a statistical behaviour [3].

In the last decade, numerical simulations and analytic arguments have contributed to clarifying the role of nonlinearities in the thermodynamic limit [4–8]. In some studies, anharmonicities have been introduced by means of self-consistent local thermostats [6, 9, 10]. Further attempts to derive Fourier's law in deterministic systems have been reported (see review papers [11, 12] and references therein). However, no rigorous derivation exists of the necessary and sufficient conditions for the validity of Fourier's law. Moreover, there are still a number of open questions concerning the steady state, such as the role of boundary conditions (BC in the following), while the convergence towards the stationary state is even less explored.

Stochastic models are rather useful in that they can effectively reproduce the evolution of deterministic nonlinear systems, while allowing for analytic solutions. Bolsterli, Rich and Visscher considered harmonic chains in which each oscillator is in contact with a stochastic thermal reservoir [9]. Then, the stationary state is obtained assuming a self-consistent condition, namely the energy current between the local reservoirs and the respective oscillator is zero. Recently, it has been proved by Bonetto *et al* that this linear model leads to a Gaussian invariant measure and the temperature profiles are linear [6]. A drawback of this model is that, strictly speaking, energy is not conserved by the bulk dynamics (see also [13] where energy current from the reservoirs becomes zero in the long time limit, and [10] for their treatment in terms of nonequilibrium Green's function formalism). Another model that can be explicitly solved is the Kipnis–Marchioro–Presutti (KMP) lattice model, in which stochastic collisions mix the energy of neighbouring particles, conserving the total energy, but not the momentum [14]. This model satisfies Fourier's law and a linear temperature profile is obtained. Energy conserving stochastic noise has also been used in lattice model systems, as natural generalizations of KMP and of the single exclusion process (SEP) [17].

Based on the KMP model, Basile *et al* have recently studied a harmonic chain in the presence of random collisions among triples of nearest-neighbour oscillators [15]. This latter process, which conserves both energy and momentum, amounts to a diffusion on the energy shell. In this system, it is proven that the energy–current autocorrelation decays as $t^{-1/2}$ and thereby heat conductivity diverges with the size of the system N, as $\kappa \sim N^{1/2}$ [15]. More recently, Jara *et al* have studied the relationship between anomalous heat transport and fractional diffusive processes [16]. They find that in the infinite system, the dynamics leading to anomalous transport is obtained from a Levy stable process that corresponds to a spacetime scaling given by the fractional diffusive operator $\partial_t - \nabla^{3/2}$, where ∇ is the gradient operator.

In a recent paper [18], we have studied a harmonic chain with both energy- and momentum-conserving noise (and fixed BC). The model, a slight variant of the one introduced in [15], is amenable to analytical calculations, as the evolution equations for the covariance matrix are linear (see also [19]). Taking a suitable continuum limit, recalled in section 2.3, we obtained a solution for the covariance matrix in the stationary state $t \to \infty$. From this solution we derived exact expressions for the temperature profile and the heat current, finding that the heat conductivity diverges as in [15]. Remarkably, the temperature profile (equation (8) in [18]) is parameter free, suggesting that it may represent a wider class of systems. To our knowledge, this is the first example of an analytic expression for the temperature profile in a system characterized by anomalous heat transport and confirms that a nonlinear shape persists in the thermodynamic limit.

In this and in the companion paper [20], we extend the previous analysis to the nonstationary case, obtaining a hydrodynamic equation ruling the relaxation of the covariance matrix towards the stationary state. More precisely, here we analytically investigate the continuum limit, by progressively eliminating the fast variables. The second part contains a detailed numerical analysis of all aspects that are not (easily) accessible to an analytic investigation, including the case of free BC, and the estimate of finite-size effects. At variance with normal heat conduction, the role of BCs is very important in the anomalous case. For instance, in disordered chains of linear oscillators, the same system may even behave as a thermal superconductor or as an insulator, by simply switching from free to fixed BC [4]. In the present context, we find that in the thermodynamic limit and given bath temperatures, fixed and free BC give rise to the same scaling behaviour, but macroscopically different values of the heat flux. Even more surprising, we find that the stationary value of the heat flux varies the coupling strength with the heat baths only for free BC.

The numerical analysis [20] demonstrates that these effects are caused by boundary layers, where the scaling behaviour of some observables changes with respect to the bulk. This phenomenon, which is associated with strong deviations from local equilibrium, hinders the development of an analytical solution in the general case. Nevertheless, for fixed BC, we demonstrate that the boundary layers do not affect the relevant physical observables, and an explicit solution can be obtained for the evolution of the temperature profile. Our results, obtained for large but finite chains, are consistent with those derived directly in the infinite-size limit [16]. This was not *a priori* granted in the presence of long-range correlations.

This paper is organized as follows. In section 2, we define the stochastic model and introduce the main notations. More precisely, in section 2.1, we introduce the covariance matrix, by adopting a different definition with respect to [18], so as to be able to treat both free and fixed BC. The corresponding ordinary differential equations are derived in sections 2.1 and 2.2, with reference to the bulk and boundaries, respectively. In section 2.3, we perform a further change of variables to simplify the treatment of the continuum limit, discussed in section 2.4. There, we define the smallness parameter, $\varepsilon = 1/\sqrt{N}$ (N is the chain length) and thereby map the discrete spatial indices i, and j of the correlation matrices onto two continuous variables (continuum limit) x and y. Moreover, we introduce an Ansatz for the scaling behaviour of the different variables, as suggested by the numerical analysis presented in the companion paper [20]. The internal consistency of the resulting equations confirms *a posteriori* the correctness of our initial choice. In section 3, we anticipate the main results to allow the reader appreciating them without being distracted by the technical details. The derivation of the partial differential equations and the elimination of the fast degrees of freedom is illustrated in sections 4 and 5, with reference to the bulk and boundary dynamics, respectively. The relaxation towards the stationary state is then discussed in section 6. The numerical computation of the spectrum of the covariance evolution operator corroborates the analytical results. Some concluding remarks are finally presented in section 7.

2. Stochastic model

We consider a homogeneous chain of N harmonic oscillators of unit mass and frequency ω. The first and Nth oscillators are coupled to Langevin heat baths at different temperatures. The equations of motion of this system are

$$\begin{aligned} \dot{q}_n &= p_n \\ \dot{p}_n &= \omega^2(q_{n+1} - 2q_n + q_{n-1}) + \delta_{n,1}(\xi^+ - \lambda\dot{q}_1) + \delta_{n,N}(\xi^- - \lambda\dot{q}_N). \end{aligned} \tag{1}$$

Here, p_n and q_n are the momentum and displacement (from its equilibrium position) of the nth oscillator and $\xi^\pm$ are independent Wiener processes with zero mean and variance $2\lambda k_B T_\pm$, where k_B is the Boltzmann constant and λ is the coupling constant. Additionally, the deterministic dynamics is perturbed by random binary collisions between nearest-neighbour oscillators occurring at a rate γ. These collisions are defined so that the total momentum and energy are conserved. This type of stochastic noise is known in the literature as *conservative noise*.

The phase-space probability density $P(\vec{q}, \vec{p}, t)$ evolves according to the equation

$$\frac{\partial P}{\partial t} = (\mathcal{L}_0 + \mathcal{L}_{\rm coll})\, P. \tag{2}$$

The first term corresponds to the usual Liouville generator of the dynamics, acting on the probability density as

$$\mathcal{L}_0 P = \sum_{i,j} \left(\mathbf{a}_{ij} \frac{\partial x_j P}{\partial x_i} + \frac{\mathbf{d}_{ij}}{2} \frac{\partial^2 P}{\partial x_i \partial x_j} \right), \tag{3}$$

with the $2N$ vector $\mathbf{x} = (q_1, q_2, \ldots, q_N, p_1, p_2, \ldots, p_N)$, and the $2N \times 2N$ matrices $\mathbf{a}$ and $\mathbf{d}$ are

$$\mathbf{a} = \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \omega^2\mathbf{g} & \lambda\mathbf{r} \end{pmatrix}; \qquad \mathbf{d} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\lambda k_B T(\mathbf{r} + \eta\mathbf{s}) \end{pmatrix}, \tag{4}$$

where T is the average temperature $(T_+ + T_-)/2$, η is the relative temperature difference $(\eta = (T_+ - T)/T)$, $\mathbf{0}$ and $\mathbf{1}$ are the null and unit $N \times N$ matrices,

$$\mathbf{r}_{ij} = \delta_{i,j}(\delta_{i,1} + \delta_{i,N}), \qquad \mathbf{s}_{ij} = \delta_{i,j}(\delta_{i,1} - \delta_{i,N}), \tag{5}$$

and $\mathbf{g}$ is the negative of the discrete Laplacian

$$\mathbf{g}_{ij} = 2\delta_{i,j} - \delta_{i+1,j} - \delta_{i,j+1}. \tag{6}$$

The stochastic collision generator is

$$\mathcal{L}_{\rm coll} P = \gamma \sum_{j=1}^{N-1} [P(\ldots, p_{j+1}, p_j, \ldots) - P(\ldots, p_j, p_{j+1}, \ldots)]. \tag{7}$$

Each term in the sum expresses the probability balance for each elementary process in which the momenta of each pair $j, j+1$ are exchanged with a rate γ.

In this paper, we consider either free or fixed boundary conditions and the choice adopted will be explicitly stated wherever needed.

2.1. Covariance matrix

The main subject of our analysis is the evolution of the covariance matrix of this system, namely the two-point correlation functions of the phase space variables. In order to develop a formalism that is both able to describe the case of fixed and free BC, we consider the correlators of relative displacements and momentum, namely $\{\Delta q_i = q_i - q_{i-1}, p_i\}$. More precisely, we study the covariance matrix

$$\mathbf{c} = \begin{pmatrix} \mathbf{y} & \mathbf{z} \\ \mathbf{z}^\dagger & \mathbf{v} \end{pmatrix}, \tag{8}$$

where the correlation matrices $\mathbf{y}, \mathbf{z}$ and $\mathbf{v}$ of respective dimensions $(N-1)\times(N-1)$, $(N-1)\times N$ and $N\times N$ are defined as

$$\mathbf{y}_{i,j} = \langle \Delta q_i \Delta q_j \rangle, \qquad \mathbf{z}_{i,j} = \langle \Delta q_i p_j \rangle, \qquad \mathbf{v}_{i,j} = \langle p_i p_j \rangle \tag{9}$$

and $\langle\cdot\rangle$ denotes the average over phase space probability distribution function P. Note that $\mathbf{y}$ and $\mathbf{v}$ are symmetric by definition, while $\mathbf{z}$ has no definite symmetry.

In terms of the relative displacements, the boundary conditions are defined by imposing

$$\Delta q_1 = 0 \qquad \text{and} \qquad \Delta q_{N+1} = 0 \tag{10}$$

for free BC and

$$\Delta q_1 = q_1 \qquad \text{and} \qquad \Delta q_{N+1} = -q_N \tag{11}$$

for fixed BC.

Using $\{\Delta q_i\}$ instead of $\{q_i\}$ is necessary since the average of q_i is not well defined when free BC are considered. This is at variance with [18], where we considered the correlators of the variables $\{q_i, p_i\}$. However, one can recover the old formulation by noticing that there exists a simple mapping with the correlators $\mathbf{U}_{i,j} = \langle q_i q_j\rangle$, $\mathbf{Z}_{i,j} = \langle q_i p_j\rangle$ and $\mathbf{V}_{i,j} = \langle p_i p_j\rangle$, studied in [18], namely

$$\mathbf{y}_{i,j} = \mathbf{U}_{i,j} - \mathbf{U}_{i,j+1} - \mathbf{U}_{i+1,j} + \mathbf{U}_{i+1,j+1}, \qquad \mathbf{z}_{i,j} = \mathbf{Z}_{i,j} - \mathbf{Z}_{i-1,j} \quad \text{and} \quad \mathbf{v}_{i,j} = \mathbf{V}_{i,j}. \tag{12}$$

The evolution equations for $\mathbf{c}$ have two contributions: the dynamic contribution directly obtained from the equations of motion (1), and the stochastic contribution that is evaluated upon multiplying (7) by $x_i x_j$ and thereby integrating over phase space.

We say that a given correlator in $\mathbf{c}$ is in the *bulk* of the system if the index of the momentum variable is in $[2, N-1]$ and the index of Δq is in $[3, N-1]$ for free BC and in $[2, N]$ for fixed BC. The evolution equations $\mathbf{c}$ are in the bulk

$$\begin{aligned}
\dot{\mathbf{y}}_{i,j} &= \mathbf{z}_{j,i} - \mathbf{z}_{j,i-1} + \mathbf{z}_{i,j} - \mathbf{z}_{i,j-1},\\
\dot{\mathbf{z}}_{i,j} &= \mathbf{v}_{i,j} - \mathbf{v}_{i-1,j} + \omega^2(\mathbf{y}_{i,j+1} - \mathbf{y}_{i,j}) + \gamma(\mathbf{z}_{i,j+1} + \mathbf{z}_{i,j-1} - 2\mathbf{z}_{i,j}),\\
\dot{\mathbf{v}}_{i,j} &= \omega^2(\mathbf{z}_{j+1,i} - \mathbf{z}_{j,i} + \mathbf{z}_{i+1,j} - \mathbf{z}_{i,j}) + \gamma W_{i,j},
\end{aligned} \tag{13}$$

where the $N\times N$ collision matrix W, corresponding to the contribution from the stochastic noise, depends on the distance between the evaluated indices i and j and can be written in a compact form as

$$\begin{aligned}
W_{ij} = {}& \tilde\delta_{i,j}[\tilde\delta_{i,j-1}(\tilde\delta_{i,N}\mathbf{v}_{i+1,j} + \tilde\delta_{j,1}\mathbf{v}_{i,j-1}) + \tilde\delta_{i,j+1}(\tilde\delta_{i,1}\mathbf{v}_{i-1,j} + \tilde\delta_{j,N}\mathbf{v}_{i,j+1})]\\
&+ \delta_{i,j}(\tilde\delta_{i,N}\mathbf{v}_{i+1,j+1} + \tilde\delta_{j,1}\mathbf{v}_{i-1,j-1}) - (2(\tilde\delta_{i,j-1} + \tilde\delta_{i,j+1} - \delta_{i,j})\\
&- \delta_{i,1} - \delta_{i,N} - \delta_{j,1} - \delta_{j,N} + \delta_{i,1}\delta_{j,1} + \delta_{i,N}\delta_{j,N})\mathbf{v}_{i,j},
\end{aligned} \tag{14}$$

where $\delta_{i,j}$ is the Kronecker delta function and $\tilde\delta_{i,j} \equiv 1 - \delta_{i,j}$. W_{ij} also holds for the boundary terms, and since it deals with momentum variables only, it is independent of the specific BC.

2.2. Boundary conditions

As a consequence of the physical boundary conditions imposed on the oscillators of the chain edges, the border terms of the covariance matrices follow a dynamics that is different from (13). The BC affect the phase space variables Δq_i, according to (10) and (11), and p_i, through the boundary terms in (14). Furthermore, the coupling between the oscillators at the edges and the heat baths affects the evolution of their momentum (1). In this section, we content ourselves with writing down the equations of motion of the border covariances corresponding to the *first column* and *last row* of each matrix. The equations for the matrix elements in the last column and first row can be obtained analogously. The latter are not a mirror image of the former. Nevertheless, one can verify that to leading order in the continuum limit, they lead to the same behaviour.

For fixed BC the equations of motion for the border matrix elements of the first matrix column (index 1 for both phase space variables) are

$$\dot{\mathbf{y}}_{i,1} = \tilde{\delta}_{i,N+1}\mathbf{z}_{1,i} - \tilde{\delta}_{i,1}\mathbf{z}_{1,i-1} + \mathbf{z}_{i,1}, \tag{15}$$

$$\dot{\mathbf{z}}_{i,1} = \tilde{\delta}_{i,N+1}\mathbf{v}_{i,1} - \tilde{\delta}_{i,1}\mathbf{v}_{i-1,1} + \omega^2(\mathbf{y}_{i,2} - \mathbf{y}_{i,1}) + \gamma(\mathbf{z}_{i,2} - \mathbf{z}_{i,1}) - \lambda\mathbf{z}_{i,1}, \tag{16}$$

$$\dot{\mathbf{v}}_{i,1} = \omega^2(\mathbf{z}_{i+1,1} - \mathbf{z}_{i,1} + \mathbf{z}_{2,i} - \mathbf{z}_{1,i}) - \lambda(1 + \delta_{i,1} + \delta_{i,N})\mathbf{v}_{i,1} + \delta_{i,1}2\lambda k_{\mathrm{B}}T_+ + \gamma W_{i,1}, \tag{17}$$

where $1 \leqslant i \leqslant N+1$ in (15), (16) and $1 \leqslant i \leqslant N$ in (17). For fixed BC the last matrix row different from zero corresponds to the index $N+1$ for Δq variables and to the index N for p variables. The corresponding equations of motion are

$$\dot{\mathbf{y}}_{N+1,j} = -\mathbf{z}_{j,N} + \tilde{\delta}_{j,N+1}\mathbf{z}_{N+1,j} - \tilde{\delta}_{j,1}\mathbf{z}_{N+1,j-1}, \tag{18}$$

$$\begin{aligned}\dot{\mathbf{z}}_{N+1,j} = -\mathbf{v}_{N,j} + \omega^2(\mathbf{y}_{N+1,j+1} - \mathbf{y}_{N+1,j}) + \gamma(\tilde{\delta}_{j,N}\mathbf{z}_{N+1,j+1} + \tilde{\delta}_{j,1}\mathbf{z}_{N+1,j-1}\\ - (\tilde{\delta}_{j,N} + \tilde{\delta}_{j,1})\mathbf{z}_{N+1,j}),\end{aligned} \tag{19}$$

$$\dot{\mathbf{v}}_{N,j} = \omega^2(\mathbf{z}_{N+1,j} - \mathbf{z}_{N,j} + \mathbf{z}_{j+1,N} - \mathbf{z}_{j,N}) - \lambda(1 + \delta_{j,1} + \delta_{j,N})\mathbf{v}_{N,j} + \delta_{j,N}2\lambda k_{\mathrm{B}}T_- + \gamma W_{N,j}, \tag{20}$$

where $1 \leqslant j \leqslant N+1$ in (18) and $1 \leqslant j \leqslant N$ in (19) and (20).

For free BC the first matrix column (different from zero) is index 2 for Δq variables and 1 for p variables. Consequently, the equations of motion are

$$\dot{\mathbf{y}}_{i,2} = \mathbf{z}_{2,i} - \mathbf{z}_{2,i-1} + \mathbf{z}_{i,2} - \mathbf{z}_{i,1}, \tag{21}$$

$$\dot{\mathbf{z}}_{i,1} = \mathbf{v}_{i,1} - \mathbf{v}_{i-1,1} + \omega^2\mathbf{y}_{i,2} + \gamma(\mathbf{z}_{i,2} - \mathbf{z}_{i,1}) - \lambda\mathbf{z}_{i,1}, \tag{22}$$

$$\dot{\mathbf{v}}_{i,1} = \omega^2(\tilde{\delta}_{i,N}\mathbf{z}_{i+1,1} - \tilde{\delta}_{i,1}\mathbf{z}_{i,1} + \mathbf{z}_{2,i}) - \lambda(1 + \delta_{i,1} + \delta_{i,N})\mathbf{v}_{i,1} + \delta_{i,1}2\lambda k_{\mathrm{B}}T_+ + \gamma W_{i,1}, \tag{23}$$

where $2 \leqslant i \leqslant N$ in (21), (22) and $1 \leqslant i \leqslant N$ in (23). For the border matrix elements of the last row (index N for both Δq and p variables), the equations of motion are

$$\dot{\mathbf{y}}_{N,j} = \mathbf{z}_{j,N} - \mathbf{z}_{j,N-1} + \mathbf{z}_{N,j} - \mathbf{z}_{N,j-1}, \tag{24}$$

$$\begin{aligned}\dot{\mathbf{z}}_{N,j} = \mathbf{v}_{N,j} - \mathbf{v}_{N-1,j} + \omega^2(\tilde{\delta}_{j,N}\mathbf{y}_{N,j+1} - \tilde{\delta}_{j,1}\mathbf{y}_{N,j}) - \lambda(\delta_{j,1} + \delta_{j,N})\mathbf{z}_{N,j}\\ + \gamma(\tilde{\delta}_{j,N}\mathbf{z}_{N,j+1} + \tilde{\delta}_{j,1}\mathbf{z}_{N,j-1} - (\tilde{\delta}_{j,1} + \tilde{\delta}_{j,N})\mathbf{z}_{N,j}),\end{aligned} \tag{25}$$

$$\dot{\mathbf{v}}_{N,j} = \omega^2(-\mathbf{z}_{N,j} - \mathbf{z}_{j,N} + \tilde{\delta}_{j,N}\mathbf{z}_{j+1,N}) - \lambda(1+\delta_{j,1} + \delta_{j,N})\mathbf{v}_{N,j} + \delta_{j,N}2\lambda k_{\mathrm{B}}T_- + \gamma W_{N,j}, \tag{26}$$

where $2 \leqslant j \leqslant N$ in (24) and $1 \leqslant j \leqslant N$ in (25), (26).

The equations presented in this section (together with the equations for the border elements of the first row and of the last column of the matrices) constitute the dynamic boundary conditions of (13).

2.3. Change of variables

We shall see in the following sections that certain combinations of covariances appear naturally in the evolution equations. It is, therefore, convenient to introduce some of these combinations as new variables. An important example of such combinations is

$$\psi_{i,j} = \mathbf{v}_{i,j} - \omega^2 \mathbf{y}_{i,j}, \tag{27}$$

which has a precise physical meaning: its diagonal elements $\psi_{i,i}$ correspond to the local balance between kinetic and potential energies. If our system satisfies the virial theorem, then we should find that $\psi_{i,i} = 0$ for all i. As we will discuss in section 5.1, this is not always the case. In the rest of this paper we will substitute $\mathbf{y}$ in favour of ψ.

In [18], we found that the stationary state solution of (13) obtained by taking all time derivatives to zero, implies that the covariance matrix $\mathbf{Z}$ is antisymmetric. From (12) it is clear that the covariance $\mathbf{z}$ has no definite symmetry, not even in the stationary state. Therefore, it is pertinent, as we do in the following, to decompose $\mathbf{z}$ into its symmetric and antisymmetric components $\mathbf{z}^{\pm}$:

$$\mathbf{z}_{i,j} = \mathbf{z}^{+}_{i,j} + \mathbf{z}^{-}_{i,j}, \qquad \text{with} \qquad \mathbf{z}^{\pm}_{i,j} = \frac{\mathbf{z}_{i,j} \pm \mathbf{z}_{j,i}}{2}. \tag{28}$$

In these variables, the equations of motion (13) are

$$\dot{\psi}_{i,j} = \omega^2\left[-4\mathbf{z}^{+}_{i,j} + \mathbf{z}^{+}_{i,j+1} + \mathbf{z}^{+}_{i+1,j} + \mathbf{z}^{+}_{i-1,j} + \mathbf{z}^{+}_{i,j-1} - \mathbf{z}^{-}_{i,j+1} + \mathbf{z}^{-}_{i+1,j} - \mathbf{z}^{-}_{i-1,j} + \mathbf{z}^{-}_{i,j-1}\right] + \gamma W_{i,j}, \tag{29a}$$

$$\begin{aligned} 2\dot{\mathbf{z}}^{-}_{i,j} = \gamma\big[&\mathbf{z}^{-}_{i,j+1} + \mathbf{z}^{-}_{i+1,j} + \mathbf{z}^{-}_{i-1,j} + \mathbf{z}^{-}_{i,j-1} + \mathbf{z}^{+}_{i,j+1} - \mathbf{z}^{+}_{i+1,j} - \mathbf{z}^{+}_{i-1,j} + \mathbf{z}^{+}_{i,j-1} - 4\mathbf{z}^{-}_{i,j}\big] \\ &+ \psi_{i+1,j} - \psi_{i,j+1} + \mathbf{v}_{i,j+1} - \mathbf{v}_{i+1,j} - \mathbf{v}_{i-1,j} + \mathbf{v}_{i,j-1}, \end{aligned} \tag{29b}$$

$$\begin{aligned} 2\dot{\mathbf{z}}^{+}_{i,j} = \gamma\big[&\mathbf{z}^{+}_{i,j+1} + \mathbf{z}^{+}_{i+1,j} + \mathbf{z}^{+}_{i-1,j} + \mathbf{z}^{+}_{i,j-1} + \mathbf{z}^{-}_{i,j+1} - \mathbf{z}^{-}_{i+1,j} - \mathbf{z}^{-}_{i-1,j} + \mathbf{z}^{-}_{i,j-1} - 4\mathbf{z}^{+}_{i,j}\big] \\ &+ 2\psi_{i,j} - \psi_{i+1,j} - \psi_{i,j+1} + \mathbf{v}_{i,j+1} + \mathbf{v}_{i+1,j} - \mathbf{v}_{i-1,j} - \mathbf{v}_{i,j-1}, \end{aligned} \tag{29c}$$

$$\dot{\mathbf{v}}_{i,j} = \omega^2\left[-2\mathbf{z}^{+}_{i,j} + \mathbf{z}^{+}_{i,j+1} + \mathbf{z}^{+}_{i+1,j} - \mathbf{z}^{-}_{i,j+1} + \mathbf{z}^{-}_{i+1,j}\right] + \gamma W_{i,j}. \tag{29d}$$

Other useful combinations of covariances will be considered when needed.

2.4. Perturbative expansion and continuum limit

Our first goal is to transform the set of difference equations (29*a*)–(29*d*) into a set of partial differential equations, by taking a continuum limit that is appropriate to our problem. We do this by following a perturbative-like analysis and choose $\varepsilon = 1/\sqrt{N}$ as a perturbation parameter. In order to proceed we first need to attribute the right order (in powers of ε) to the covariance matrix elements. In order to keep the presentation as simple as possible, we proceed on the basis of our knowledge of the stationary solution [18] and of the numerical solutions discussed in [20].

In the stationary state, $\mathbf{y}_{i,j}$ and $\mathbf{v}_{i,j}$ are $O(\varepsilon)$, with the exception of the diagonal terms that are $O(1)$, being proportional to the mean potential and kinetic energy, respectively. On the other hand, the combination in equation (27) is $O(\varepsilon^2)$, indicating that the system is locally at equilibrium. Moreover, it can be shown that the fields $\mathbf{z}^{\pm}$ are x-derivatives of $\mathbf{Z}$, namely $\mathbf{z}^{+} \propto \mathbf{Z}_x$ and $\mathbf{z}^{-} \propto \mathbf{Z}_{xx}$. In [18], we found that off-diagonal terms of $\mathbf{Z}$ are $O(1)$, while along the diagonal $\mathbf{Z} = 0$ due to its antisymmetry. Since each differentiation wrt x increases the order by ε (see (32) below), $\mathbf{z}^+$ must be $O(\varepsilon)$ and $\mathbf{z}^-$ of $O(\varepsilon^2)$. Altogether,

$$
\begin{aligned}
\psi_{i,j} &\equiv \varepsilon^2\delta_{i,j}\Omega_i + \varepsilon^2(1-\delta_{i,j})\Psi_{i,j},\\
\mathbf{z}^+_{i,j} &\equiv \varepsilon\delta_{i,j}\mathcal{S}^+_i + \varepsilon(1-\delta_{i,j})\mathcal{Z}^+_{i,j},\\
\mathbf{z}^-_{i,j} &\equiv \varepsilon^2(1-\delta_{i,j})\mathcal{Z}^-_{i,j},\\
\mathbf{v}_{i,j} &\equiv \delta_{i,j}T_i + \varepsilon(1-\delta_{i,j})\mathcal{V}_{i,j}.
\end{aligned}
\tag{30}
$$

These equations also fix the notation that we will use in the rest of the paper. Note that in this notation, all the correlators appearing on the rhs of the above definitions, namely $\Omega, \Psi, \mathcal{S}^+, \mathcal{Z}^+, \mathcal{Z}^-, T$ and $\mathcal{V}$, are $O(1)$. We refer the reader to [20], where we give numerical support for the validity of (30). Furthermore, we have found in [20] that for fixed BC, the off-diagonal elements of ψ scale as ε^3 for $\lambda = \omega$, and as in (30) for $\lambda \neq \omega$. In what follows we will assume (30) and discuss the modifications for the resonant case $\lambda = \omega$ where appropriate.

The last step before proceeding to the derivation of the partial differential equations consists in transforming the discrete indices of the correlators into two suitable continuous variables. We do this by introducing the variable y for the diagonal direction and variable x for the transversal direction as (see, e.g., figure 1 in [18])

$$
x \equiv (i-j)\varepsilon; \qquad y \equiv \frac{(i+j)\varepsilon^2 - 1}{1-|i-j|\varepsilon^2}. \tag{31}
$$

The nonlinear definition of y is chosen so that its domain $[-1, 1]$ is independent of x. Nevertheless, in the limit $N \to \infty$, the effects of the nonlinearities are localized at the boundaries of the domain and thus do not complicate the study of the bulk dynamics. Differential changes in these variables can be written as

$$
x' = x + f\varepsilon; \qquad y' = \frac{(i+j)\varepsilon^2 - 1 + s\varepsilon^2}{1-|i-j|\varepsilon^2 - f\varepsilon^2}, \tag{32}
$$

where the integer *shift* functions $f(\Delta i, \Delta j) \equiv \Delta i - \Delta j$ and $s(\Delta i, \Delta j) \equiv \Delta i + \Delta j$ ($f, s : \mathbb{Z}^2 \mapsto \mathbb{Z}$) account for the displacements on the discrete variables along the x and y directions, respectively.

Using this, the continuum limit of any covariance matrix element $\mathbf{m}_{i+\Delta i, j+\Delta j}$ can be written up to $O(\varepsilon^2)$ as

$$
\mathbf{m}_{i+\Delta i, j+\Delta j} = \mathbf{m}(x + f\varepsilon, y + \varepsilon^2(fy + s)), \qquad \text{for} \quad i \geqslant j, \tag{33}
$$

for the continuous correlator function $\mathbf{m}$. Here and in what follows, we keep the same notation for the continuous functions derived from the matrix variables (e.g. $\mathbf{v}_{i,j}(t) \longrightarrow \mathbf{v}(x, y, t)$, etc). If we restrict to the matrix *lower triangle*, as we do in (33) and in the following, then in the limit $N \to \infty$ the variables (x, y) belong to the domain

$$
\mathcal{D} \equiv \{(x, y) | x \in [0, \infty); y \in [-1, 1]\}. \tag{34}
$$

Note, for instance, that $x = \text{const}$ corresponds to moving along the diagonal direction ($x = 0$ corresponding to the main diagonal). For more details we refer the reader to section 4.1 of [18].

3. Main results

In the next section we show that, after eliminating the fast degrees of motion, the bulk dynamics reduces to the following two equations (the subscript denoting partial derivation):

$$
\dot{\mathcal{Z}}^+ = \varepsilon^2 \left(\gamma \mathcal{Z}^+_{xx} + 2\mathcal{V}_y\right), \tag{35}
$$

$$\dot{\mathcal{V}} = \varepsilon^2 \left(\gamma \mathcal{V}_{xx} + 2\omega^2 \mathcal{Z}^+_y\right), \tag{36}$$

together with the conditions on the boundary of the domain (34)

$$\delta\mathcal{D} \equiv \{x = 0\} \cup \{y = \pm 1\}.$$

On $x = 0$, we find that

$$\mathcal{V}_x(0, y, t) = 0, \tag{37}$$

$$\mathcal{Z}^+_x(0, y, t) = -\frac{1}{\gamma} T_y, \tag{38}$$

$$\dot{T}(y, t) = 2\varepsilon^3\omega^2\mathcal{Z}^+_y(0, y, t). \tag{39}$$

This last equation is accompanied by its BC, namely $T(\mp 1, t) = T_\pm$. The peculiar mathematical structure of the problem should be underlined: the boundary evolution of $\mathcal{Z}$ is determined dynamically by the differential equations (38) and (39) which, in turn, determine the temperature field $T(y, t)$.

On $y = \pm 1$ the BC are particularly difficult because close to these boundaries the expansions (30) are no longer valid. The numerical solution in [20] shows that a kind of 'boundary layers' (BL) exist, namely that at least some of the fields scale differently in the regions of $\mathcal{D}$ lying within a distance $O(\varepsilon)$ from $y = \pm 1$. In particular, in this region, ψ is of order $O(\varepsilon)$. Recalling that ψ is the difference between the kinetic and potential energy, which away from the diagonal are $O(\varepsilon)$, this implies that in the BL the relative difference between potential and kinetic terms is of $O(1)$, which means that the system is not in a local thermal equilibrium.

The existence of a BL hinders us from finding an explicit exact solution in the general case. This technical difficulty is irrelevant in two cases: (a) for fixed BC and (b) for free BC at the 'resonant' value of the bath coupling constant, $\lambda = \omega$. In case (a), the boundary layer does not affect the relevant fields and, as we show in section 5, it is sufficient to impose

$$\mathcal{Z}^+(x, \pm 1, t) = 0, \qquad \mathcal{V}(x, \pm 1, t) = 0. \tag{40}$$

This allows us to find an explicit time-dependent solution. From a physical point of view, case (b) corresponds to the only situation in which the coupling can be perfectly tuned to avoid an impedance mismatch on the boundaries. Unfortunately, we were not able to find an explicit solution in this case.

Inspection of the equations of motion reveals that the temperature field T is the slowest variable (it evolves on a time scale $O(\varepsilon^{-3})$). Since $\mathcal{Z}^+$ and $\mathcal{V}$ relax on a time scale $O(\varepsilon^{-2})$ they can be adiabatically eliminated by setting their time derivative equal to zero in equations (35) and (36). By further eliminating the field $\mathcal{V}$, we obtain the fourth-order equation

$$\gamma^2\mathcal{Z}^+_{xxxx} - 4\omega^2\mathcal{Z}^+_{yy} = 0, \tag{41}$$

that is formally equal to the equation solved in [18], the main difference being that here $\mathcal{Z}^+$ is time dependent. The dynamical equation is obtained by first determining the stationary solution of (41) with the appropriate BC and then using (38) to express $\mathcal{Z}^+$ as a function of T and replacing the result into (39). This is accomplished by considering the Fourier expansion

$$T(y, t) = T_s(y) + \sum_{n=1}^{\infty} \mathcal{T}_n(t) \sin\left[\frac{n\pi}{2}(y + 1)\right], \tag{42}$$

with T_s being the stationary solution of T, as given by formulae (18) and (19) of [18]. In section 6 we show that the coefficients $\mathcal{T}_n(t)$ obey the linear equation (74). The associated eigenvalues, that must be computed numerically as the problem is not exactly diagonal, uniquely determine the relaxation to the steady state for any assigned initial condition. They are found to be proportional to $-(k/N)^{3/2}$ (for k being a positive integer labelling the eigenvalues). This is reminiscent of the spectrum of the eigenvalues of the fractional Laplacian $\nabla^{3/2}$, thus suggesting that the evolution of the temperature field is ruled by some underlying fractional diffusion equation on hydrodynamic scales [16, 21].

4. Dynamical equations

We first focus on the dynamics in the 'bulk' of system, namely the interior of $\mathcal{D}$. In appendix A we derive the set of coupled partial differential equations (A.3*a*)–(A.3*d*), for the covariances $\psi, \mathbf{z}^-, \mathbf{z}^+$ and $\mathbf{v}$. Furthermore, using (30) the equations with their explicit order in ε are rewritten as

$$\dot{\Psi} = 2\varepsilon\left(2\omega^2\mathcal{Z}^-_x + \omega^2\mathcal{Z}^+_{xx} + \gamma\mathcal{V}_{xx}\right) + 4\varepsilon^2\omega^2 y\mathcal{Z}^-_y, \tag{43a}$$

$$\dot{\mathcal{Z}}^- = \varepsilon\Psi_x + \varepsilon^2\left(\gamma\mathcal{Z}^-_{xx} + y\Psi_y\right), \tag{43b}$$

$$\dot{\mathcal{Z}}^+ = \varepsilon^2\left(\gamma\mathcal{Z}^+_{xx} + 2\mathcal{V}_y\right) - \varepsilon^3\left(\tfrac{1}{2}\Psi_{xx} + \Psi_y\right), \tag{43c}$$

$$\dot{\mathcal{V}} = \varepsilon^2\left(2\omega^2\mathcal{Z}^-_x + \omega^2\mathcal{Z}^+_{xx} + 2\gamma\mathcal{V}_{xx} + 2\omega^2\mathcal{Z}^+_y\right) + 2\varepsilon^3\omega^2 y\mathcal{Z}^-_y. \tag{43d}$$

The four variables can be split into a pair of *fast* (Ψ and $\mathcal{Z}^-$) and *slow* ($\mathcal{Z}^+$ and $\mathcal{V}$) ones, which evolve on time scales of orders ε^{-1} and ε^{-2}, respectively. Upon substituting the x-derivative of (43*a*) into the time derivative of (43*b*), we find

$$\ddot{\mathcal{Z}}^- - 4\varepsilon^2\omega^2\mathcal{Z}^-_{xx} - 2\varepsilon^2\left(\omega^2\mathcal{Z}^+_{xxx} + \gamma\mathcal{V}_{xxx}\right) = \varepsilon^2\gamma\dot{\mathcal{Z}}^-_{xx} + 8\varepsilon^3\omega^2 y\mathcal{Z}^-_{xy} + 2\varepsilon^3 y\left(\omega^2\mathcal{Z}^+_{xxy} + \mathcal{V}_{xxxy}\right), \tag{44}$$

where we have retained only terms up to order ε^3. The terms on the rhs do not affect the final solution but must be taken into account to justify the adiabatic elimination. In fact, scaling the time by ε, we see that they are $o(\varepsilon^3)$ and could, in principle, be neglected. However, if we do so, we are left with a non-dissipative wave equation (with source terms) for $\mathcal{Z}^-$, that cannot account for the convergence towards the steady state. Therefore, to study the evolution of the fast variables, we are obliged to include the higher order terms (losses are actually provided by the $\dot{\mathcal{Z}}^-_{xx}$ term, the other being perturbations of the source term). After this remark, we are authorized to adiabatically eliminate $\mathcal{Z}^-$ and Ψ. From (43*a*), (43*b*) we obtain to leading order

$$\mathcal{Z}^-_x = -\frac{1}{2}\mathcal{Z}^+_{xx} + \frac{\gamma}{2\omega^2}\mathcal{V}_{xx}, \tag{45a}$$

$$\Psi_x = 0. \tag{45b}$$

Note that Ψ is a constant moving away from the diagonal. We now turn our attention to the slow variables and substitute the above two equations into (43*c*), (43*d*). To leading order, we finally obtain (35) and (36).

At this point, it is useful to illustrate some features of (35) and (36). The symmetry of these equations suggests to introduce the new variables $\mathcal{Q}^{(\pm)} = \omega\mathcal{Z}^+ \pm \mathcal{V}$ that allow decoupling the system of equations into

$$\dot{\mathcal{Q}}^{(\pm)} = \varepsilon^2\left(\gamma\mathcal{Q}^{(\pm)}_{xx} \pm 2\omega\mathcal{Q}^{(\pm)}_y\right). \tag{46}$$

The first term on the rhs describes a transversal diffusion process characterized by the diffusion constant $\varepsilon^2\gamma$; the second term accounts for a longitudinal right/left sound-wave propagation, depending on whether $\mathcal{Q}^-/\mathcal{Q}^+$ is considered. Leaving aside for the moment the issue of BC, by absorbing the order ε^2 in the time variable, we look for solutions of the form

$$\mathcal{Q}^+(x, y, t) = P(x, t) \cos K_y y + Q(x, t) \sin K_y y; \tag{47}$$

we consider only $\mathcal{Q}^+$ as the equation for $\mathcal{Q}^-$ leads to the same dispersion relation. By substituting this in (46) and separating the independent terms, we obtain

$$\dot{P} = \gamma P_{xx} + 2\omega K_y Q, \tag{48}$$

$$\dot{Q} = \gamma Q_{xx} - 2\omega K_y P. \tag{49}$$

By then assuming the following form of the solution:

$$P(x, t) = a(t) \cos K_x x + b(t) \sin K_x x, \tag{50}$$

$$Q(x, t) = c(t) \cos K_x x + d(t) \sin K_x x, \tag{51}$$

we find that $a(t)$ and $c(t)$ satisfy a system of two ordinary differential equations:

$$\dot{a} = -\gamma K_x^2 a - 2\omega K_y c, \tag{52}$$

$$\dot{c} = -\gamma K_x^2 c + 2\omega K_y a \tag{53}$$

(b and d do not add any information as they satisfy the same set of equations). Looking for solutions of the form $a(t) = \tilde{a} \exp(\mu t)$, $c(t) = \tilde{c} \exp(\mu t)$, the resulting eigenvalue equation yields two degenerate branches for the dispersion relations

$$\mu = -\gamma K_x^2 \pm 2\omega \mathrm{i} K_y, \tag{54}$$

where i denotes the imaginary unit.

Taking $\mu = 0$, we recover the eigenvalue of the stationary state [18]. Most importantly, the real part of μ is negative, thus ensuring the stability of the stationary state. By reintroducing the ε^2 factor in the time units, we can thus conclude that modes characterized by a K_x of O(1) relax on a time scale of order ε^{-2}. As discussed in [20], these are the modes that mostly contribute to the relevant nonzero off-diagonal correlations. However, K_x can, by construction, be as small as ε^{-1}. As a result the slowest relaxation times that one can observe are of the order of N^2. This is indeed confirmed by the numerical calculation of the spectrum of the evolution operator [20].

5. Boundary conditions

A complete solution of the dynamical problem requires solving the set of partial differential equations (43*a*)–(43*d*) on the whole domain $\mathcal{D}$ (34), including its boundary at all times. In a general context, the difficulty of obtaining a full solution depends on the constraints along $\delta\mathcal{D}$. If they amount to algebraic conditions or if the boundary dynamics is faster than the bulk dynamics, then one deals with the standard type of *static* BC. In the opposite case, it is the bulk dynamics that can be adiabatically eliminated and the relevant (long-term) evolution would be controlled by what happens along the boundaries.

In the first section, we study the dynamics of the covariance elements close to the diagonal and derive a differential equation describing the evolution of the temperature profile. Later, in section 5.2, we study how the physical BC determine the dynamics of the correlators along the boundaries $y = \pm 1$.

5.1. Boundary conditions along $x = 0$

The BC along the diagonal are not connected with the physical BC of the chain, but rather with the symmetry of the various matrices. In appendix B, we derive the differential equations describing the evolution of the covariances at $x = 0$ (B.3*a*)–(B.3*c*) and (B.5*a*)–(B.5*d*). As the diagonal terms Ω and $\mathcal{S}^+$ appear only as differences with their respective off-diagonal counterparts, it is convenient to introduce $\delta\Psi = \Psi - \Omega$ and $\varepsilon\delta\mathcal{Z}^+(y,t) = \mathcal{Z}^+(0,y,t) - \mathcal{S}^+(y,t)$, where we benefit of the numerical investigations to anticipate that the latter difference is of higher order than the single addenda[2].

By introducing $\delta\Psi$ and $\delta\mathcal{Z}^+$ and using (30), we obtain the following set of partial differential equations describing the evolution at $x = 0$:

$$\delta\dot\Psi = -2\omega^2\left(3\delta\mathcal{Z}^+ + \mathcal{Z}^- + 2\mathcal{Z}^+_x\right) + 2\gamma\mathcal{V}_x + \varepsilon\left(-4\omega^2 y\mathcal{Z}^+_y + \gamma\mathcal{K}_\mathcal{V}\right), \tag{55a}$$

$$\dot\Psi = 2\omega^2(\mathcal{Z}^- - \delta\mathcal{Z}^+) + 2\gamma\mathcal{V}_x + \varepsilon\left(2\omega^2\left(\mathcal{Z}^+_{xx} + 2\mathcal{Z}^-_x\right) + \gamma\mathcal{K}_\mathcal{V}\right), \tag{55b}$$

$$\dot{\mathcal{Z}}^- = \frac{\delta\Psi}{2} - \gamma\mathcal{Z}^- + T_y + \varepsilon(\Psi_x - \mathcal{V}_y), \tag{55c}$$

$$\delta\dot{\mathcal{Z}}^+ = \frac{3}{2}\delta\Psi - \gamma\left(3\delta\mathcal{Z}^+ + 2\mathcal{Z}^+_x\right) + T_y + \varepsilon\left(\Psi_x - 2\gamma y\mathcal{Z}^+_y - \mathcal{V}_y\right), \tag{55d}$$

$$\dot{\mathcal{Z}}^+ = \varepsilon\left(\frac{1}{2}\delta\Psi - \gamma\delta\mathcal{Z}^+ + T_y\right) + \varepsilon^2\left(\mathcal{V}_y + \gamma\mathcal{Z}^+_{xx}\right), \tag{55e}$$

$$\dot{\mathcal{V}} = \varepsilon(\omega^2(\mathcal{Z}^- - \delta\mathcal{Z}^+) + 2\gamma\mathcal{V}_x) + \varepsilon^2\left(\omega^2\left(\mathcal{Z}^+_{xx} + 2\mathcal{Z}^-_x + 2\mathcal{Z}^+_y\right) + \gamma\mathcal{K}_\mathcal{V}\right), \tag{55f}$$

$$\dot T = 2\varepsilon^2\omega^2\left(\delta\mathcal{Z}^+ + \mathcal{Z}^- + \mathcal{Z}^+_x\right) + \varepsilon^3\omega^2\left(\mathcal{Z}^+_{xx} + 2\mathcal{Z}^-_x + 2(y+1)\mathcal{Z}^+_y\right), \tag{55g}$$

where we have considered the first two leading contributions to the evolution of each variable and, for the sake of compactness, we have introduced

$$\mathcal{K}_\mathcal{V} = 3\mathcal{V}_{xx} + 2y\mathcal{V}_y. \tag{56}$$

Like in the bulk, the variables evolve over manifestly different time scales, and the temperature field T is the slowest one. Therefore, we proceed by adiabatically eliminating the other variables, starting from the fastest ones. By setting the first three time derivatives equal to zero and considering only the leading terms, equations (55*a*), (55*b*) and (55*c*) lead to

$$\delta\Psi = -\left(\gamma\mathcal{Z}^+_x + \frac{\gamma^2}{\omega^2}\mathcal{V}_x + 2T_y\right), \tag{57a}$$

$$\mathcal{Z}^- = -\frac{1}{2}\left(\frac{\gamma}{\omega^2}\mathcal{V}_x + \mathcal{Z}^+_x\right), \tag{57b}$$

$$\delta\mathcal{Z}^+ = \frac{1}{2}\left(\frac{\gamma}{\omega^2}\mathcal{V}_x - \mathcal{Z}^+_x\right). \tag{57c}$$

Moreover, the stationary solution of (55*d*) yields

$$\gamma\mathcal{Z}^+_x + \frac{3\gamma^2}{2\omega^2}\mathcal{V}_x + T_y = 0. \tag{58}$$

By now inserting (57*a*) and (57*c*) into (55*e*), and setting the time derivative $\dot{\mathcal{Z}}^+$ equal to zero, we obtain (37). This equation has an obvious meaning: $\mathcal{V}$ is symmetric by definition across the diagonal, so that we naturally expect $\mathcal{V}$ to be maximal for $x = 0$.

[2] As a matter of fact, assigning to $\delta\mathcal{Z}^+$ the same scaling as its addenda leads to *unphysical* super fast evolution.

Furthermore, by using (37) in (58), we obtain the second relevant constraint (38). From (57*b*) and (57*c*), one can easily find that $\mathcal{Z}^- = \delta\mathcal{Z}^+$. By then referring back to (57*a*), (57*b*) and (57*c*), and using constraints (37) and (38), we obtain

$$\delta\mathcal{Z}^+ = \mathcal{Z}^- = \frac{1}{2\gamma}T_y, \qquad \delta\Psi = -T_y. \tag{59}$$

Altogether, once the five conditions contained in (37), (38) and (59) are satisfied, it turns out that the derivatives of all seven boundary variables (55*a*)–(55*g*) are equal to zero to leading order. In particular, we see that the variable Ψ remains undetermined, but this is not a problem, as it is of higher order in ε and does not contribute to the leading-order evolution of the physically relevant variables. Additionally, with the exception of $\mathcal{V}$, all the variables are expressed as a function of the temperature profile T, whose evolution must be determined if we want to find a closed solution.

It turns out that the leading contribution of $\mathrm{O}(\varepsilon^2)$ to $\dot{T}$ is zero. Therefore, it is necessary to go one order further in the perturbative analysis. This can be easily done by noting that the leading contribution to $\dot{T}$ is equal to that of $\dot{\Omega} = \dot{\Psi} - \delta\dot{\Psi}$ (see (B.3*a*)). Subtracting (55*b*) from (55*a*) and setting the time derivatives equal to zero we find that, up to $\mathrm{O}(\varepsilon^2)$,

$$2\omega^2\big(\delta\mathcal{Z}^+ + \mathcal{Z}^- + \mathcal{Z}^+_x\big) = -\varepsilon\omega^2\big(\mathcal{Z}^+_{xx} + 2\mathcal{Z}^-_x + 2y\mathcal{Z}^+_y\big). \tag{60}$$

By inserting this into (55*g*) and retaining terms up to $\mathrm{O}(\varepsilon^3)$, we obtain equation (39). By recalling that $\mathcal{Z}^+_y$ is proportional to the divergence of the heat flux along the diagonal, we recognize that (39) is nothing but the continuity equation for the energy and could have been derived simply on the basis of physical arguments. However, the relaxation of the temperature profile towards the stationary state occurs on a time scale that is $\mathrm{O}(\varepsilon^{-3})$, i.e., for $t \sim N^{3/2}$. As a consequence, we can conclude that the bulk dynamics is faster than that occurring along the diagonal and can, thereby, be adiabatically eliminated as anticipated in section 3.

Finally, in order to complete the treatment, we must complement (39) with its physical BC, as it is a (one-dimensional) partial differential equation. Without the need of a formal treatment, it is easily understood that these BC are simply $T(\pm 1, t) = T_\mp$. As a matter of fact, the relaxation on the boundaries occurs on a finite time scale ($\approx 1/\lambda$), i.e. it is basically instantaneous with respect to the above-mentioned time scales.

5.2. *Boundary conditions along* $y = \pm 1$

In this section we analyse the conditions that the physical boundary conditions, either fixed or free, impose on the covariances. At the chain edges, where the system is directly coupled to the heat baths stochastic evolution, the dynamics is different from that in the bulk: on the one hand, the deterministic restoring force is not counterbalanced by the boundary and on the other hand, the stochastic collision for the edge oscillators is also a 'one-sided' process. Consequently, we introduce new auxiliary variables to distinguish the boundary dynamics from its bulk counterpart. More precisely, we define $\phi_{i,1} \equiv \psi_{i,1}$, $\zeta^\pm_{i,1} \equiv \mathbf{z}^\pm_{i,1}$ and $\nu_{i,1} \equiv \mathbf{v}_{i,1}$.

In appendix C we derive the partial differential equations describing the dynamics of the covariance at $y = -1$ for fixed and free BC. However, in this case we are not entitled to use (30) in order to assign the correct order in ε of the covariance variables. As we have discussed in section 3, there exist a boundary layer, namely a region around $y = \pm 1$ of size ε^{-1}, where the scaling of the covariance matrix on ε differs from (30). This BL has been further studied numerically in [20]. It is important to note that if we insist using (30) then mathematical consistency requires that the order of, e.g., ψ is $\mathrm{O}(\varepsilon)$ and not $\mathrm{O}(\varepsilon^2)$, which is also what we numerically observe in [20]. However, (30) cannot differentiate the scalings in the BL from those in the bulk. Consequently, in this section we do not use the expansions (30) and limit ourselves to extract some physical information from the leading contribution determined only by the differential structure of the equations. First we focus on fixed BC.

For fixed BC we end up with four equations for the bulk variables and four equations for the boundary variables (C.3*a*)–(C.3*h*). Since all equations evolve on time scales of O(1), they can be adiabatically eliminated from the bulk dynamics, which is at least $O(\varepsilon)$. By taking all time derivatives to zero and solving the resulting set of relations, we find that in the stationary state all boundary variables coincide with the respective bulk counterparts

$$\zeta^+ = \mathbf{z}^+, \qquad \zeta^- = \mathbf{z}^-, \qquad \nu = \mathbf{v} \quad \text{and} \quad \phi = \psi, \tag{61}$$

and that

$$\mathbf{v}(x, -1, t) = 0, \tag{62}$$

$$\mathbf{z}^+(x, -1, t) = -\mathbf{z}^-(x, -1, t). \tag{63}$$

It is interesting to point out that these relations are independent of the parameters of the system and that (61) imply that the bulk variables are continuous at the boundaries. Recalling that in (63) $\mathbf{z}^-$ is of higher order than its symmetric counterpart $\mathbf{z}^+$, we do obtain (40). It is interesting to note that the indetermination of $\psi(x, -1)$ and $\mathbf{z}^-(x, -1)$, on which the effects of the BL are mostly observed, does not affect the leading order dynamical solution of the physically relevant fields. Therefore, for fixed BC, the existence of a BL does not impede us from using the boundary relations (40) and determining the evolution of $\mathbf{v}$ and $\mathbf{z}^+$.

We now turn our attention to free BC. As seen in appendix C.2, in this case, only two auxiliary variables along the boundaries are necessary, $\zeta_{i,1} \equiv \mathbf{z}_{i,1}$, corresponding to the non symmetrizable term, and $\nu_{i,1} \equiv \mathbf{v}_{i,1}$. From (C.5*a*)–(C.5*d*) we find that when the three conditions

$$\nu = \mathbf{v}, \qquad \tilde{\zeta} = \mathbf{z}^+ \quad \text{and} \quad \mathbf{z}^- = 0 \tag{64}$$

are satisfied, the four time derivatives are equal to zero (to leading order), thus satisfying the stationary state solution. By using these relations in (C.5*e*)–(C.5*f*), we obtain the relevant mathematical conditions

$$\psi(x, -1, t) = \mathbf{v}(x, -1, t) - \lambda \mathbf{z}^+(x, -1, t), \tag{65}$$

$$\omega^2 \mathbf{z}^+(x, -1, t) = \lambda \mathbf{v}(x, -1, t). \tag{66}$$

There are two main differences between (62), (63) and the equations above: first, (65), (66) depend on the variable ψ, which, as we have discussed, is the variable whose behaviour is most affected by the BL. Second and more important, the free BC relations now depend on the parameters λ and ω.

By combining (65) and (66), we find that

$$(\omega^2 - \lambda^2)\mathbf{z}^+(x, -1, t) = \lambda \psi(x, -1, t). \tag{67}$$

If $\omega = \lambda$, then $\psi(x, -1) = 0$ (or at least $O(\varepsilon^2)$), consistently with our expectations from the bulk dynamics. In this resonant case the boundary condition reduces to $\omega \mathcal{Z}^+(x, -1, t) = \mathcal{V}(x, -1, t)$. Though simple, we have not found a way to derive an explicit solution of the bulk equation which satisfies this constraint.

In the non-resonant regime $\omega^2 \neq \lambda^2$, (67) implies that $\mathbf{z}^+$ and ψ are of the same order along the boundary. The numerical studies presented in [20] confirm this prediction, but show also that this is because ψ is of $O(\varepsilon)$. Such an observation is seemingly inconsistent with the bulk analysis which predicts ψ to be of higher order. As said, the only way to solve the paradox is by invoking the presence of a BL connecting the two different scaling regimes.

Summarizing this section, the BC for $y = \pm 1$ reveal a crucial difference between fixed BC and free BC. In the former case they are independent of parameters of the system, which in turn imply that the asymptotic profile as well as the leading term of the heat flux is *universal*. In the latter case both quantities depend on the coupling strength with the heat baths λ. In this very atypical situation the contact with the baths may lead to measurable macroscopic effects. Note, in particular, that the heat flux for a system with anomalous thermal conductivity, like the one that concerns us, in general may depend on the type of boundary conditions, even in the infinite volume limit.

6. Dynamics of the temperature field

The evolution of the temperature profile is determined by (39), subjected to conditions (37) and (38). Since the temperature field evolves on a slower time scale, the bulk dynamics can be adiabatically eliminated. Therefore, it suffices to solve (41) for $\mathcal{Z}^+$ and plug its solution into (39). As discussed above, an explicit calculation is feasible only for fixed BC, where the first of conditions (40) implies that we can, following [18], expand the time-dependent solution of $\mathcal{Z}^+$ as

$$\mathcal{Z}^+(x, y, t) = \sum_n B_n(x, t) \sin\left[\frac{n\pi}{2}(y + 1)\right]. \tag{68}$$

The Fourier coefficients satisfy the ordinary differential equation

$$\frac{\partial^4 B_n}{\partial x^4} = -\left(\frac{n\pi\omega}{\gamma}\right)^2 B_n, \tag{69}$$

whose explicit solution yields

$$B_n(x, t) = A_n(t) \exp(-\alpha_n x) \sin(\alpha_n x) + A_n'(t) \exp(-\alpha_n x) \cos(\alpha_n x), \tag{70}$$

where $\alpha_n \equiv \sqrt{n\pi\omega/2\gamma}$ and we have discarded the components which diverge for $x \to \infty$. By differentiating the equilibrium solution of (35) with respect to x, we realize that the condition (37) is equivalent to $\mathcal{Z}^+_{xxx}(0, y, t) = 0$, which in turn implies that $A_n = -A_n'$.

If one is interested only in the stationary solution, equation (39) implies that $\mathcal{Z}^+(0, y) =$ constant, namely that the heat flux is constant along the chain. This condition transforms itself into distinct equations for the coefficients $\{A_n\}$, which can therefore be determined (apart from a multiplicative factor). Afterwards, with the help of (38) we can determine $T_s(y)$. The unknown multiplicative and additive factors are eventually removed by imposing $T_s(\mp 1) = T_\pm$. We do not report these calculations as they would closely follow what already reported in [18].

Here, we wish to solve the dynamical problem, particularly for the temperature field $T(y, t)$. Let us consider its Fourier expansion (42) where we have only included the terms that are appropriate for fixed BC. To write down closed equations for the coefficients $\mathcal{T}_n$, we must face the problem that the two sides of equation (39) are expanded in a different set of functions, namely sines and cosines, respectively, and the problem is therefore not diagonal. By using vector notations with an obvious meaning of the symbols, we obtain from (38) and (39)

$$\mathcal{A} = \frac{1}{2\gamma}\mathbf{D}\mathbf{R}\mathcal{T} \tag{71}$$

$$\dot{\mathcal{T}} = 2\varepsilon^3\omega^2\mathbf{R}\mathcal{A}, \tag{72}$$

where

$$R_{n,k} = \begin{cases} 2k^2/(k^2 - n^2) & \text{for} \quad k + n \quad \text{odd} \\ 0 & \text{otherwise} \end{cases} \tag{73}$$

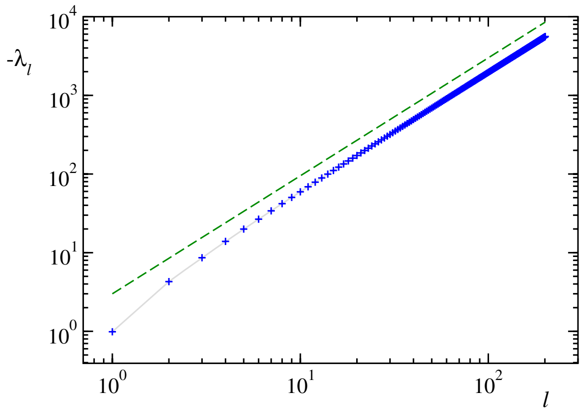

Figure 1. Spectrum $\{\lambda_\ell\}$ of the linear equation (74). The eigenvalues are expressed in $\varepsilon^3\omega^2/\gamma$ time units. The straight line corresponds to a power law with a rate 3/2.

and $D_{n,m} = \delta_{nm}/\alpha_n$. We can thus write a closed equation for $\mathcal{T}$,

$$\dot{\mathcal{T}} = \frac{\varepsilon^3\omega^2}{\gamma}\,\mathbf{RDR}\mathcal{T}. \tag{74}$$

A numerical evaluation reveals that **RDR** is almost diagonal. In fact, the eigenvectors are very close to Fourier sine-modes [20]. In figure 1, we report the eigenvalues in ascending order, versus the index ℓ that is equal (within a proportionality factor) to the corresponding wave number. The data align almost perfectly along a straight line (in log–log scales) that corresponds to a scaling with a power $3/2$. The deviations observed at small wavenumbers are not due to the truncation of the operator **RDR**; they express the fact that **RDR** is intrinsically defined on a finite domain. In [20], we show that the numerical solution of the entire dynamical operator (without any approximation) confirms our analytical predictions.

Altogether, the spacetime scaling of (74) indicates that the evolution of the temperature field $T(y, t)$ is, on the considered time-scales, ruled by a diffusion equation with a fractional Laplacian $\nabla^{3/2}$. Recently, this has been shown to be the case for a similar model system [16], directly in the infinite-N limit (i.e. without including the effect of the boundary conditions).

7. Discussion and conclusions

We have presented a detailed description of the relaxation towards the nonequilibrium steady state in a model of harmonic oscillators with conservative noise. To our knowledge, this is an almost unique instance where relaxation phenomena can be studied in great detail in a realistic setup. By implementing the continuum-limit ideas previously introduced in [18] we have obtained a set of partial differential equations describing the evolution of the covariance matrix. In the bulk, the velocity–velocity and the symmetric component of the velocity–position correlations are the relevant (slow) variables: they appear to evolve on a time scale of order $1/\varepsilon^2 = N$. This means that in the bulk, relaxation phenomena are mostly controlled by the propagation of sound waves.

Along the boundaries, the evolution of the relevant two-point correlators can be explicitly determined to the lowest order in ε. Again, these correlators evolve on different time scales, the temperature field being the slowest one (its dynamical equation evolves on time scales

$t \sim N^{3/2}$). By adiabatically eliminating the fast variables, we find that $T(y, t)$ satisfies an energy continuity equation, where the expression for the current can be obtained from the stationary solution of the bulk equation. Altogether, the temperature field $T(y, t)$ appears to satisfy a diffusion equation with a fractional Laplacian $\nabla^{3/2}$. However, the relationship with fractional Brownian motion should be further explored. In particular, it is not yet clear as to what extent the temperature profile can be obtained from the solution of the fractional equation in a finite domain.

The case of free BC remains open in view of the difficulties arising along the boundaries, where the mathematical conditions depend explicitly on system parameters such as ω and λ. This is not only the indication of a lack of 'universality' but implies also that some variables (namely ψ) must scale differently in the bulk and along the boundaries. As a result, boundary layers are expected to arise (and this is confirmed by the numerical analysis carried out in the companion paper [20]) which would require a separate analysis. This is one of the open problems that will be worth investigating in the future, especially in the perspective that a similar scenario might hold in generic nonlinear deterministic systems. Only in the resonant case $\omega^2 = \lambda^2$, boundary layers do not exist. However, even in this limit, the mathematical conditions holding on the boundaries are sufficiently complicate to prevent the derivation of explicit expressions (at least, to the best of our knowledge).

All of our analysis has been *restricted* to two-point correlators. The main reason is that the dynamical equations are exactly closed onto themselves, so that there is no need to invoke higher order correlators. Moreover, this analysis allows determining exact expressions for the most relevant variables such as the heat flux and the temperature profile. However, one should not forget that the scaling behaviour of heat conductivity in this stochastic model ($\kappa \simeq N^{1/2}$) differs from that of generic nonlinear systems, where $\kappa \simeq N^{1/3}$. Is this an indication that a faithful description of such systems needs including higher-order correlators? More modestly, it would be already interesting to check to what extent a Gaussian approximation of the invariant measure based on the knowledge of two-point correlators can accurately describe other variables such as, e.g., energy fluctuations.

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Appendix A. Derivation of the differential equations: bulk dynamics

In this appendix we derive the set of partial differential equations describing the dynamics of the covariance matrix elements in the bulk of the system, namely for covariances $\mathbf{m}_{i,j}$ for which

(i) the index of the momentum variable is in $[2, N-1]$,
(ii) the index of Δq is in $[3, N-1]$ for free BC and in $[2, N]$ for fixed BC, and
(iii) $|i-j| > 1$.

In this situation the stochastic collision matrix is simply

$$W_{ij} = \mathbf{v}_{i+1,j} + \mathbf{v}_{i,j-1} + \mathbf{v}_{i-1,j} + \mathbf{v}_{i,j+1} - 4\mathbf{v}_{i,j}. \tag{A.1}$$

In the continuum limit, discrete shifts of the indices i and j yield infinitesimal changes of the field variables x and y and by using (33), the set of difference equations (29*a*)–(29*d*) lead to a set of continuous equation given explicitly by

$$\begin{aligned}\dot{\psi}(x,y) = \omega^2[&-4\mathbf{z}^+(x,y) + \mathbf{z}^+(x-\varepsilon, y-\varepsilon^2(y-1)) + \mathbf{z}^+(x+\varepsilon, y+\varepsilon^2(y+1))\\ &+ \mathbf{z}^+(x-\varepsilon, y-\varepsilon^2(y+1)) + \mathbf{z}^+(x+\varepsilon, y+\varepsilon^2(y-1)) - \mathbf{z}^-(x-\varepsilon, y-\varepsilon^2(y-1))\\ &+ \mathbf{z}^-(x+\varepsilon, y+\varepsilon^2(y+1)) - \mathbf{z}^-(x-\varepsilon, y-\varepsilon^2(y+1)) + \mathbf{z}^-(x+\varepsilon, y+\varepsilon^2(y-1))]\\ &+ \gamma[\mathbf{v}(x+\varepsilon, y+\varepsilon^2(y+1)) + \mathbf{v}(x-\varepsilon, y-\varepsilon^2(y-1)) + \mathbf{v}(x-\varepsilon, y-\varepsilon^2(y+1))\\ &+ \mathbf{v}(x+\varepsilon, y+\varepsilon^2(y-1)) - 4\mathbf{v}(x,y)],\end{aligned} \tag{A.2a}$$

$$\begin{aligned}2\dot{\mathbf{z}}^-(x,y) = \gamma[&-4\mathbf{z}^-(x,y) + \mathbf{z}^-(x-\varepsilon, y-\varepsilon^2(y-1)) + \mathbf{z}^-(x+\varepsilon, y+\varepsilon^2(y+1))\\ &+ \mathbf{z}^-(x-\varepsilon, y-\varepsilon^2(y+1)) + \mathbf{z}^-(x+\varepsilon, y+\varepsilon^2(y-1)) + \mathbf{z}^+(x-\varepsilon, y-\varepsilon^2(y-1))\\ &- \mathbf{z}^+(x+\varepsilon, y+\varepsilon^2(y+1)) - \mathbf{z}^+(x-\varepsilon, y-\varepsilon^2(y+1)) + \mathbf{z}^+(x+\varepsilon, y+\varepsilon^2(y-1))]\\ &+ \psi(x+\varepsilon, y+\varepsilon^2(y+1)) - \psi(x-\varepsilon, y-\varepsilon^2(y-1)) + \mathbf{v}(x-\varepsilon, y-\varepsilon^2(y-1))\\ &- \mathbf{v}(x+\varepsilon, y+\varepsilon^2(y+1)) - \mathbf{v}(x-\varepsilon, y-\varepsilon^2(y+1)) + \mathbf{v}(x+\varepsilon, y+\varepsilon^2(y-1)),\end{aligned} \tag{A.2b}$$

$$\begin{aligned}2\dot{\mathbf{z}}^+(x,y) = \gamma[&-4\mathbf{z}^+(x,y) + \mathbf{z}^+(x-\varepsilon, y-\varepsilon^2(y-1)) + \mathbf{z}^+(x+\varepsilon, y+\varepsilon^2(y+1))\\ &+ \mathbf{z}^+(x-\varepsilon, y-\varepsilon^2(y+1)) + \mathbf{z}^+(x+\varepsilon, y+\varepsilon^2(y-1)) + \mathbf{z}^-(x-\varepsilon, y-\varepsilon^2(y-1))\\ &- \mathbf{z}^-(x+\varepsilon, y+\varepsilon^2(y+1)) - \mathbf{z}^-(x-\varepsilon, y-\varepsilon^2(y+1)) + \mathbf{z}^-(x+\varepsilon, y+\varepsilon^2(y-1))]\\ &+ 2\psi(x,y) - \psi(x+\varepsilon, y+\varepsilon^2(y+1)) - \psi(x-\varepsilon, y-\varepsilon^2(y-1))\\ &+ \mathbf{v}(x-\varepsilon, y-\varepsilon^2(y-1)) + \mathbf{v}(x+\varepsilon, y+\varepsilon^2(y+1))\\ &- \mathbf{v}(x-\varepsilon, y-\varepsilon^2(y+1)) - \mathbf{v}(x+\varepsilon, y+\varepsilon^2(y-1)),\end{aligned} \tag{A.2c}$$

$$\begin{aligned}\dot{\mathbf{v}}(x,y) = \omega^2[&-2\mathbf{z}^+(x,y) + \mathbf{z}^+(x-\varepsilon, y-\varepsilon^2(y-1)) + \mathbf{z}^+(x+\varepsilon, y+\varepsilon^2(y+1))\\ &- \mathbf{z}^-(x-\varepsilon, y-\varepsilon^2(y-1)) + \mathbf{z}^-(x+\varepsilon, y+\varepsilon^2(y+1))] + \gamma[\mathbf{v}(x+\varepsilon, y+\varepsilon^2(y+1))\\ &+ \mathbf{v}(x-\varepsilon, y-\varepsilon^2(y-1)) + \mathbf{v}(x-\varepsilon, y-\varepsilon^2(y+1)) + \mathbf{v}(x+\varepsilon, y+\varepsilon^2(y-1))\\ &- 4\mathbf{v}(x,y)].\end{aligned} \tag{A.2d}$$

Finally, the straightforward differentiation in the continuous coordinates x and y, up to $O(\varepsilon^2)$, lead to a set of partial differential equations for the time evolution of these four correlators:

$$\dot{\psi} = 4\varepsilon\omega^2\mathbf{z}^-_x + 2\varepsilon^2\left(\omega^2\left(\mathbf{z}^+_{xx} + 2y\mathbf{z}^-_y\right) + \gamma\mathbf{v}_{xx}\right), \tag{A.3a}$$

$$\dot{\mathbf{z}}^- = \varepsilon\psi_x + \varepsilon^2\left(\gamma\mathbf{z}^-_{xx} + y\psi_y\right), \tag{A.3b}$$

$$\dot{\mathbf{z}}^+ = \varepsilon^2\left(-\tfrac{1}{2}\psi_{xx} + \gamma\mathbf{z}^+_{xx} - \psi_y + 2\mathbf{v}_y\right), \tag{A.3c}$$

$$\dot{\mathbf{v}} = 2\varepsilon\omega^2\mathbf{z}^-_x + \varepsilon^2\left(\omega^2\left(\mathbf{z}^+_{xx} + 2y\mathbf{z}^-_y + 2\mathbf{z}^+_y\right) + 2\gamma\mathbf{v}_{xx}\right). \tag{A.3d}$$

Appendix B. Derivation of the differential equations: $x = 0$

In this appendix, we derive a set of partial differential equations for the evolution of the diagonal covariances. The dynamics in the boundary $\{x = 0\}$ is different from the bulk dynamics due to the stochastic collisions and is not related to the *physical* boundary conditions concerning the coupling with the heat baths.

The dynamics along this boundary are obtained by considering the difference equations (29*a*)–(29*d*) along the diagonal ($i = j$), and along the the sub-diagonal ($i - j = 1$). Along the diagonal, the difference equations become

$$\dot{\Omega}_i = 2\omega^2\left(-2\mathbf{s}^+_{i,i} + \mathbf{z}^+_{i+1,i} + \mathbf{z}^+_{i,i-1} + \mathbf{z}^-_{i+1,i} + \mathbf{z}^-_{i,i-1}\right) + \gamma(T_{i-1} + T_{i+1} - 2T_i), \tag{B.1a}$$

$$\dot{\mathbf{s}}^+_i = \gamma\left(-2\mathbf{s}^+_i + \mathbf{z}^+_{i+1,i} + \mathbf{z}^+_{i,i-1} - \mathbf{z}^-_{i+1,i} + \mathbf{z}^-_{i,i-1}\right) + (\Omega_i - \psi_{i+1,i} + \mathbf{v}_{i+1,i} - \mathbf{v}_{i,i-1}), \tag{B.1b}$$

$$\dot{T}_i = 2\omega^2\left(-\mathbf{s}^+_i + \mathbf{z}^+_{i+1,i} + \mathbf{z}^-_{i+1,i}\right) + \gamma(T_{i-1} + T_{i+1} - 2T_i). \tag{B.1c}$$

The continuum limit rule (33) for the diagonal on sub-diagonal matrix elements can be written as

$$\mathbf{m}_{i+\Delta i,i+\Delta j} = \mathbf{m}(f\varepsilon, y + \varepsilon^2(fy + s)), \tag{B.2}$$

with the shift functions f and s defined as before. By using this rule and differentiating the resulting continuous equations, we obtain up to $O(\varepsilon^2)$

$$\dot{\Omega} = 4\omega^2\left(\mathbf{z}^+ - \mathbf{s}^+ + \mathbf{z}^-\right) + 4\varepsilon\omega^2\left(\mathbf{z}^+_x + \mathbf{z}^-_x\right) + 2\varepsilon^2\omega^2\left(\mathbf{z}^+_{xx} + \mathbf{z}^-_{xx} + 2y\left(\mathbf{z}^+_y + \mathbf{z}^-_y\right)\right), \tag{B.3a}$$

$$\begin{aligned}\dot{\mathbf{s}}^+ &= \Omega - \psi + 2\gamma(\mathbf{z}^+ - \mathbf{s}^+) + \varepsilon\left(-\psi_x + 2\gamma\mathbf{z}^+_x\right)\\ &\quad + \varepsilon^2\left(-\tfrac{1}{2}\psi_{xx} + \gamma\mathbf{z}^+_{xx} - (y+1)\psi_y + 2\gamma\left(y\mathbf{z}^+_y - \mathbf{z}^-_y\right) + 2\mathbf{v}_y\right),\end{aligned} \tag{B.3b}$$

$$\dot{T} = 2\omega^2(\mathbf{z}^+ - \mathbf{s}^+ + \mathbf{z}^-) + 2\varepsilon\omega^2\left(\mathbf{z}^+_x + \mathbf{z}^-_x\right) + \varepsilon^2\omega^2\left(\mathbf{z}^+_{xx} + \mathbf{z}^-_{xx} + 2(y+1)\left(\mathbf{z}^+_y + \mathbf{z}^-_y\right)\right). \tag{B.3c}$$

Analogously, on the lower diagonal ($i - j = 1$), the difference equations become

$$\begin{aligned}\dot{\psi}_{i,i-1} &= \omega^2\left(-4\mathbf{z}^+_{i,i-1} + \mathbf{s}^+_i + \mathbf{z}^+_{i+1,i-1} + \mathbf{s}^+_{i-1} + \mathbf{z}^+_{i,i-2} + \mathbf{z}^-_{i+1,i-1} + \mathbf{z}^-_{i,i-2}\right)\\ &\quad + \gamma(\mathbf{v}_{i+1,i-1} + \mathbf{v}_{i,i-2} - 2\mathbf{v}_{i,i-1}),\end{aligned} \tag{B.4a}$$

$$\begin{aligned}2\dot{\mathbf{z}}^-_{i,i-1} &= \gamma\left(-4\mathbf{z}^-_{i,i-1} + \mathbf{z}^-_{i+1,i-1} + \mathbf{z}^-_{i,i-2} + \mathbf{s}^+_i - \mathbf{z}^+_{i+1,i-1} - \mathbf{s}^+_{i-1,i-1} + \mathbf{z}^+_{i,i-2}\right)\\ &\quad + \psi_{i+1,i-1} - \Omega_i + T_i - \mathbf{v}_{i+1,i-1} - T_{i-1} + \mathbf{v}_{i,i-2},\end{aligned} \tag{B.4b}$$

$$\begin{aligned}2\dot{\mathbf{z}}^+_{i,i-1} &= \gamma\left(-4\mathbf{z}^+_{i,i-1} + \mathbf{s}^+_i + \mathbf{z}^+_{i+1,i-1} + \mathbf{s}^+_{i-1} + \mathbf{z}^+_{i,i-2} - \mathbf{z}^-_{i+1,i-1} + \mathbf{z}^-_{i,i-2}\right)\\ &\quad + 2\psi_{i,i-1} - \psi_{i+1,i-1} - \Omega_i + T_i + \mathbf{v}_{i+1,i-1} - T_{i-1} - \mathbf{v}_{i,i-2},\end{aligned} \tag{B.4c}$$

$$\dot{\mathbf{v}}_{i,i-1} = \omega^2\left(-2\mathbf{z}^+_{i,i-1} + \mathbf{s}^+_i + \mathbf{z}^+_{i+1,i-1} + \mathbf{z}^-_{i+1,i-1}\right) + \gamma(\mathbf{v}_{i+1,i-1} + \mathbf{v}_{i,i-2} - 2\mathbf{v}_{i,i-1}), \tag{B.4d}$$

and by using (B.2) and keeping differential terms up to $O(\varepsilon^2)$, we arrive at

$$\begin{aligned}\dot{\psi} &= 2\omega^2(-\mathbf{z}^+ + \mathbf{s}^+ + \mathbf{z}^-) + 2\varepsilon\left(2\omega^2\mathbf{z}^-_x + \gamma\mathbf{v}_x\right) + \varepsilon^2\left(2\omega^2\left(\mathbf{z}^+_{xx} + 2\mathbf{z}^-_{xx} + \mathbf{z}^+_y - \mathbf{s}^+_y\right.\right.\\ &\quad \left.\left. + (2y-1)\mathbf{z}^-_y\right) + \gamma(3\mathbf{v}_{xx} + 2y\mathbf{v}_y)\right),\end{aligned} \tag{B.5a}$$

$$\dot{\mathbf{z}}^- = \tfrac{1}{2}(\psi - \Omega) - \gamma\mathbf{z}^- + \varepsilon\psi_x + \varepsilon^2\left(\psi_{xx} + y\psi_y - \mathbf{v}_y + T_y + \gamma\left(\mathbf{z}^-_{xx} - \mathbf{z}^+_y + \mathbf{s}^+_y + \mathbf{z}^-_y\right)\right), \tag{B.5b}$$

$$\dot{\mathbf{z}}^+ = \tfrac{1}{2}(\psi - \Omega) - \gamma(\mathbf{z}^+ - \mathbf{s}^+) + \varepsilon^2\left(-\tfrac{1}{2}\psi_{xx} - \psi_y + \mathbf{v}_y + T_y + \gamma\left(\mathbf{z}^+_{xx} + \mathbf{z}^+_y - \mathbf{s}^+_y - \mathbf{z}^-_y\right)\right), \tag{B.5c}$$

$$\begin{aligned}\dot{\mathbf{v}} &= \omega^2(-\mathbf{z}^+ + \mathbf{s}^+ + \mathbf{z}^-) + 2\varepsilon\left(\omega^2\mathbf{z}^-_x + \gamma\mathbf{v}_x\right)\\ &\quad + \varepsilon^2\left(\omega^2\left(\mathbf{z}^+_{xx} + 2\mathbf{z}^-_{xx} + 2\mathbf{z}^+_y + 2y\mathbf{z}^-_y\right) + \gamma\left(3\mathbf{v}_{xx} + 2y\mathbf{v}_y\right)\right).\end{aligned} \tag{B.5d}$$

The set of partial differential equations (B.3*a*)–(B.3*c*) and (B.5*a*)–(B.5*d*) describe the dynamics of the covariance on the line $x = 0$.

Appendix C. Derivation of the differential equations: $y = \pm 1$

Along the boundaries, where the system is coupled with the heat baths, the dynamics is different from that in the bulk not only because of the coupling with the bath itself but also because the deterministic force felt by the edge oscillators feel from the bulk is not counterbalanced and, moreover, the stochastic collision at the edges is a 'one-sided' process. Clearly, the dynamics depend on whether the BC are free or fixed.

In this section we derive the covariance dynamic equations at the boundaries $y = \pm 1$. These boundaries correspond, on the original matrix, to the matrix elements of the first and last rows and columns. Restricted to the semi-infinite plane $x > 0$, i.e., to the domain (34), $y = -1$ corresponds to the first matrix column, while $y = 1$ corresponds to the last matrix row. Here we specialize on the boundary $y = -1$. The BC at $y = 1$, that can be obtained analogously, lead to the same information that is extracted from the BC at $y = -1$.

C.1. Fixed boundary conditions: $y = -1$

We recall that fixed BC are defined by the relations $\Delta q_1 = q_1$ and $\Delta q_{N+1} = -q_N$. Our starting point is transforming the set of difference equations (15)–(17) into a set on the variables Ψ, $\mathbf{z}^{\pm}$ and $\mathbf{v}$. Note, however, that the domain of Δq_i, which is $i \in [1, N+1]$, is different from the domain of p_i, $i \in [1, N]$. As a consequence, the terms $\mathbf{z}_{N+1,j}$ are nonsymmetrizable.

In what follows, we restrict to $i \in [3, N-1]$, where all terms are symmetrizable and all covariances are 'far enough' from the diagonal. By finally denoting $\phi_{i,1} \equiv \psi_{i,1}$, $\zeta^{\pm}_{i,1} \equiv \mathbf{z}^{\pm}_{i,1}$ and $\nu_{i,1} \equiv \mathbf{v}_{i,1}$, we obtain

$$\begin{aligned}\dot{\phi}_{i,1} = \omega^2\big[-4\zeta^+_{i,1} + \zeta^+_{i+1,1} + \zeta^+_{i-1,1} + \zeta^-_{i+1,1} - \zeta^-_{i-1,1} + \mathbf{z}^+_{i,2} - \mathbf{z}^-_{i,2}\big] - \lambda\nu_{i,1} \\ + \gamma[\nu_{i+1,1} + \nu_{i-1,1} + \mathbf{v}_{i,2} - 3\nu_{i,1}],\end{aligned} \tag{C.1a}$$

$$\begin{aligned}2\dot{\zeta}^-_{i,1} = \gamma\big[-3\zeta^-_{i,1} + \zeta^-_{i+1,1} + \zeta^-_{i-1,1} + \zeta^+_{i,1} - \zeta^+_{i+1,1} - \zeta^+_{i-1,1} + \mathbf{z}^-_{i,2} + \mathbf{z}^+_{i,2}\big] \\ - \lambda\big[\zeta^+_{i,1} + \zeta^-_{i,1}\big] + \phi_{i+1,1} - \psi_{i,2} - \nu_{i+1,1} - \nu_{i-1,1} + \mathbf{v}_{i,2},\end{aligned} \tag{C.1b}$$

$$\begin{aligned}2\dot{\zeta}^+_{i,1} = \gamma\big[-3\zeta^+_{i,1} + \zeta^+_{i+1,1} + \zeta^+_{i-1,1} + \zeta^-_{i,1} - \zeta^-_{i+1,1} - \zeta^-_{i-1,1} + \mathbf{z}^+_{i,2} + \mathbf{z}^-_{i,2}\big] \\ - \lambda\big[\zeta^+_{i,1} + \zeta^-_{i,1}\big] + 2\phi_{i,1} + \phi_{i+1,1} - \psi_{i,2} + \nu_{i+1,1} - \nu_{i-1,1} + \mathbf{v}_{i,2},\end{aligned} \tag{C.1c}$$

$$\begin{aligned}\dot{\nu}_{i,1} = \omega^2\big[-2\zeta^+_{i,1} + \zeta^+_{i+1,1} + \zeta^-_{i+1,1} + \mathbf{z}^+_{i,2} - \mathbf{z}^-_{i,2}\big] - \lambda\nu_{i,1} \\ + \gamma[\nu_{i+1,1} + \nu_{i-1,1} + \mathbf{v}_{i,2} - 3\mathbf{v}_{i,2}],\end{aligned} \tag{C.1d}$$

$$\begin{aligned}\dot{\psi}_{i,2} = \omega^2\big[-4\mathbf{z}^+_{i,2} + \mathbf{z}^+_{i,3} + \mathbf{z}^+_{i+1,2} + \mathbf{z}^+_{i-1,2} + \zeta^+_{i,1} - \mathbf{z}^-_{i,3} + \mathbf{z}^-_{i+1,2} - \mathbf{z}^-_{i-1,2} + \zeta^-_{i,1}\big] \\ + \gamma\big[\mathbf{v}_{i+1,2} + \mathbf{v}_{i,3} + \mathbf{v}_{i-1,2} + \nu_{i,1} - 4\mathbf{v}_{i,2}\big],\end{aligned} \tag{C.1e}$$

$$\begin{aligned}2\dot{\mathbf{z}}^-_{i,2} = \gamma\big[-4\mathbf{z}^-_{i,2} + \mathbf{z}^-_{i,3} + \mathbf{z}^-_{i+1,2} + \mathbf{z}^-_{i-1,2} + \zeta^-_{i,1} + \mathbf{z}^+_{i,3} - \mathbf{z}^+_{i+1,2} - \mathbf{z}^+_{i-1,2} + \zeta^+_{i,1}\big] \\ + \psi_{i+1,2} - \psi_{i,3} + \mathbf{v}_{i,3} - \mathbf{v}_{i+1,2} - \mathbf{v}_{i-1,2} + \nu_{i,1},\end{aligned} \tag{C.1f}$$

$$\begin{aligned}2\dot{\mathbf{z}}^+_{i,2} = \gamma\big[-4\mathbf{z}^+_{i,2} + \mathbf{z}^+_{i,3} + \mathbf{z}^+_{i+1,2} + \mathbf{z}^+_{i-1,2} + \zeta^+_{i,1} + \mathbf{z}^-_{i,3} - \mathbf{z}^-_{i+1,2} - \mathbf{z}^-_{i-1,2} + \zeta^-_{i,1}\big] \\ + 2\psi_{i,2} - \psi_{i+1,2} - \psi_{i,3} + \mathbf{v}_{i,3} + \mathbf{v}_{i+1,2} - \mathbf{v}_{i-1,2} - \nu_{i,1},\end{aligned} \tag{C.1g}$$

$$\dot{\mathbf{v}}_{i,2} = \omega^2\big[-2\mathbf{z}^+_{i,2} + \mathbf{z}^+_{i,3} + \mathbf{z}^+_{i+1,2} - \mathbf{z}^-_{i,3} + \mathbf{z}^-_{i+1,2}\big] + \gamma[\mathbf{v}_{i+1,2} + \mathbf{v}_{i,3} + \mathbf{v}_{i-1,2} + \nu_{i,1} - 4\mathbf{v}_{i,2}]. \tag{C.1h}$$

In the continuum limit, the column index 1 corresponds to $y = -1$. As a result, (33) becomes

$$\mathbf{m}_{i+\Delta i,1+\Delta j} = \mathbf{m}(x + f\varepsilon, -1 + \varepsilon^2(-f + s)), \tag{C.2}$$

with the shift functions f and s defined as before. Using this and proceeding as in appendix A, the dynamics at $y = -1$ for fixed BC is described by the following set of partial differential equations:

$$\dot{\phi} = \omega^2(\mathbf{z}^+ - \mathbf{z}^- - 2\zeta^+) - \lambda\nu - \gamma(\nu - \mathbf{v}) + \varepsilon\left[-\omega^2\mathbf{z}_x^+ - \gamma\mathbf{v}_x\right], \quad (C.3a)$$

$$2\dot{\zeta}^- = \gamma(\mathbf{z}^+ + \mathbf{z}^- - \zeta^+ - \zeta^-) - \lambda(\zeta^+ + \zeta^-) - \psi + \phi - 2\nu + \mathbf{v} - \varepsilon\left[\gamma\mathbf{z}_x^+ - \mathbf{v}_x\right], \quad (C.3b)$$

$$2\dot{\zeta}^+ = \gamma(\mathbf{z}^+ + \mathbf{z}^- - \zeta^+ - \zeta^-) - \lambda(\zeta^+ + \zeta^-) - \psi + \phi + \mathbf{v} + \varepsilon\left[-\gamma\mathbf{z}_x^+ + 2\nu_x - \mathbf{v}_x\right], \quad (C.3c)$$

$$\dot{\nu} = \omega^2(\mathbf{z}^+ - \mathbf{z}^- - \zeta^+ + \zeta^-) - \lambda\nu - \gamma(\nu - \mathbf{v}) + \varepsilon\left[\omega^2\left(\zeta_x^+ - \mathbf{z}_x^+\right) - \gamma\mathbf{v}_x\right], \quad (C.3d)$$

$$\dot{\psi} = \omega^2(-\mathbf{z}^+ - \mathbf{z}^- + \zeta^+ + \zeta^-) + \gamma(\nu - \mathbf{v}) + 4\varepsilon\omega^2\mathbf{z}_x^- + 2\varepsilon^2\left[\omega^2\mathbf{z}_{xx}^+ + \gamma\mathbf{v}_{xx}\right], \quad (C.3e)$$

$$2\dot{\mathbf{z}}^- = \gamma(-\mathbf{z}^+ - \mathbf{z}^- + \zeta^+ + \zeta^-) + \nu - \mathbf{v} + 2\varepsilon\psi_x, \quad (C.3f)$$

$$2\dot{\mathbf{z}}^+ = \gamma(-\mathbf{z}^+ - \mathbf{z}^- + \zeta^+ + \zeta^-) - \nu + \mathbf{v} + 2\varepsilon^2\left[\gamma\mathbf{z}_{xx}^+ + 2\mathbf{v}_y\right], \quad (C.3g)$$

$$\dot{\mathbf{v}} = \gamma(\nu - \mathbf{v}) + 2\varepsilon\omega^2\mathbf{z}_x^- + \varepsilon^2\left[\omega^2\left(\mathbf{z}_{xx}^+ + 2\mathbf{z}_y^+\right) + 2\gamma\mathbf{v}_{xx}\right]. \quad (C.3h)$$

C.2. Free boundary conditions: $y = -1$

We start by recalling that free BC are defined by setting $\Delta q_1 = \Delta q_{N+1} = 0$. This means that in this case, the domain of the phase variables is $i \in [2, N]$ for Δq_i and $i \in [1, N]$ for p_i. Consequently, while $\mathbf{z}_{i>1,1}$ is well defined, its symmetric component $\mathbf{z}_{1,i}$ is not, thus restricting the symmetrization of the covariance $\mathbf{z}$ (28). At variance with the case of fixed BC, here it is necessary to consider non-symmetrizable terms. In analogy to the previous section, we distinguish boundary covariances from their bulk counterparts by defining $\nu_{i,1} \equiv \mathbf{v}_{i,1}$ and $\zeta_{i,1} \equiv \mathbf{z}_{i,1}$, recalling that $\zeta_{i,1}$ is non-symmetrizable. Moreover, note that for free BC, ψ has no boundary component, as the dynamics of $\psi_{i,2}$ corresponds to that in the bulk (see (21)).

By transforming the set of difference equations (21)–(23) into the covariance variables of section 2.3 and restricting to $i \in [4, N-1]$, we obtain

$$\dot{\psi}_{i,2} = \omega^2\left[-4\mathbf{z}_{i,2}^+ + \mathbf{z}_{i,3}^+ + \mathbf{z}_{i+1,2}^+ + \mathbf{z}_{i-1,2}^+ - \mathbf{z}_{i,3}^- + \mathbf{z}_{i+1,2}^- - \mathbf{z}_{i-1,2}^- + \zeta_{i,1}\right] + \gamma[\mathbf{v}_{i+1,2} + \mathbf{v}_{i,3} + \mathbf{v}_{i-1,2} + \nu_{i,1} - 4\mathbf{v}_{i,2}], \quad (C.4a)$$

$$2\dot{\mathbf{z}}_{i,2}^- = \gamma\left[-4\mathbf{z}_{i,2}^- + \mathbf{z}_{i,3}^- + \mathbf{z}_{i+1,2}^- + \mathbf{z}_{i-1,2}^- + \mathbf{z}_{i,3}^+ - \mathbf{z}_{i+1,2}^+ - \mathbf{z}_{i-1,2}^+ + \zeta_{i,1}\right] + \psi_{i+1,2} - \psi_{i,3} + \mathbf{v}_{i,3} - \mathbf{v}_{i+1,2} - \mathbf{v}_{i-1,2} + \nu_{i,1}, \quad (C.4b)$$

$$2\dot{\mathbf{z}}_{i,2}^+ = \gamma\left[-4\mathbf{z}_{i,2}^+ + \mathbf{z}_{i,3}^+ + \mathbf{z}_{i+1,2}^+ + \mathbf{z}_{i-1,2}^+ + \mathbf{z}_{i,3}^- - \mathbf{z}_{i+1,2}^- - \mathbf{z}_{i-1,2}^- + \zeta_{i,1}\right] + 2\psi_{i,2} - \psi_{i+1,2} - \psi_{i,3} + \mathbf{v}_{i,3} + \mathbf{v}_{i+1,2} - \mathbf{v}_{i-1,2} - \nu_{i,1}, \quad (C.4c)$$

$$\dot{\mathbf{v}}_{i,2} = \omega^2\left[-2\mathbf{z}_{i,2}^+ + \mathbf{z}_{i,3}^+ + \mathbf{z}_{i+1,2}^+ - \mathbf{z}_{i,3}^- + \mathbf{z}_{i+1,2}^-\right] + \gamma[\mathbf{v}_{i+1,2} + \mathbf{v}_{i,3} + \mathbf{v}_{i-1,2} + \nu_{i,1} - 4\mathbf{v}_{i,2}]. \quad (C.4d)$$

$$\dot{\zeta}_{i,1} = \gamma\left[\mathbf{z}_{i,2}^+ + \mathbf{z}_{i,2}^- - \zeta_{i,1}\right] - \lambda\zeta_{i,1} - \psi_{i,2} + \nu_{i,1} - \nu_{i-1,1} + \mathbf{v}_{i,2}, \quad (C.4e)$$

$$\dot{\nu}_{i,1} = \omega^2\left[\zeta_{i+1,1} - \zeta_{i,1} + \mathbf{z}_{i,2}^+ - \mathbf{z}_{i,2}^-\right] - \lambda\nu_{i,1} + \gamma[\nu_{i+1,1} + \nu_{i-1,1} + \mathbf{v}_{i,2} + -3\nu_{i,1}]. \quad (C.4f)$$

Furthermore, using (C.2), we obtain the continuous version of the equations above. Differentiating these latter equations and keeping terms up to $O(\varepsilon^2)$, we obtain the following set of partial differential equations:

$$\dot{\psi} = \omega^2(-\mathbf{z}^- + \zeta - \mathbf{z}^+) + \gamma(\nu - \mathbf{v}) + 4\varepsilon\omega^2\mathbf{z}_x^- + 2\varepsilon^2\left[\omega^2\mathbf{z}_{xx}^+ + \gamma\mathbf{v}_{xx}\right], \quad (C.5a)$$

$$2\dot{\mathbf{z}}^- = \gamma(-\mathbf{z}^- + \zeta - \mathbf{z}^+) + \nu - \mathbf{v} + 2\varepsilon\psi_x, \tag{C.5b}$$

$$2\dot{\mathbf{z}}^+ = \gamma(-\mathbf{z}^- + \zeta - \mathbf{z}^+) - \nu + \mathbf{v} + 2\varepsilon^2\left(\omega^2\mathbf{z}^+_{xx} + 2\mathbf{v}_y\right), \tag{C.5c}$$

$$\dot{\mathbf{v}} = \gamma(\nu - \mathbf{v}) + 2\varepsilon\omega^2\mathbf{z}^-_x + \varepsilon^2\left[\omega^2\left(\mathbf{z}^+_{xx} + 2\mathbf{z}^+_y\right) + 2\gamma\mathbf{v}_{xx}\right], \tag{C.5d}$$

$$\dot{\zeta} = -\gamma(-\mathbf{z}^- + \zeta - \mathbf{z}^+) - \lambda\zeta + \mathbf{v} - \psi + \varepsilon\left[-\gamma\left(\mathbf{z}^+_x + \mathbf{z}^-_x\right) + \nu_x - \mathbf{v}_x + \psi_x\right], \tag{C.5e}$$

$$\dot{\nu} = \omega^2(\mathbf{z}^+ - \mathbf{z}^-) - \gamma(\nu - \mathbf{v}) - \lambda\nu + \varepsilon\left[\omega^2\left(\mathbf{z}^-_x + \zeta_x - \mathbf{z}^+_x\right) - \gamma\mathbf{v}_x\right]. \tag{C.5f}$$

We remark that the covariance's equations at $y = -1$ for the diagonal terms, namely those obtained for $i = 1$ and $i = 2$, yield the same solution to leading order, with the additional constraint $\nu_{i,1} = T_+$.

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